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COMPLETENESS AND NONUNIQUENESS OF GENERAL SOLUTIONS OF TRANSVERSELY ISOTROPIC ELASTICITY

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Abstract—In this paper we give general solutions of transversely isotropic elasticity. Their completeness and nonuniqueness are proved. We point out that famous Lekhnitskii–Hu–Nowacki solutions and Elliott–Lodge solutions are complete if the elastic region is z-convex.

1. INTRODUCTION

As Dundurs (1970) shows, one of the characteristics of mechanics of composite materials is anisotropy. Transversely isotropic materials are a noticeable kind of anisotropic material. Lekhnitskii (1940, 1981) gave general solutions of axisymmetric problems of transversely isotropic elasticity. Eubanks and Sternberg (1954) proved that Lekhnitskii's solutions are complete if the meridional half-section is z-convex. Recently, Wang *et al.* (1994) pointed out that this condition of z-convex is unnecessary for the completeness of Lekhnitskii's solutions. Hu (1953) and Nowacki (1954) extended Lekhnitskii's solutions to general three-dimensional transversely isotropic elasticity. Elliott (1948) and Lodge (1955) obtained other general solutions. Some studies for transversely isotropic elasticity can be found in Chen (1966), Alexsandrov and Soloviev (1978), Pan and Chou (1979), Okumura (1987), Zureick and Eubanks (1988), Ding and Xu (1988) and Horgan and Simmonds (1991).

It is the purpose of this paper to present a systematic method for the derivation of general solutions and the proof of their completeness in transversely isotropic elasticity. However, the general solutions are not unique. The scope of nonuniqueness of the general solutions is also given. It is the nonuniqueness that brings us advantages which allow us to prove that the famous Lekhnitskii–Hu–Nowacki solutions and Elliott–Lodge solutions are special cases of our general solutions and are complete if the elastic region is z-convex.

The method used in this paper is obtained by extending our previous work (Wang, 1981; Wang, 1985; Wang and Xu, 1990; Wang and Wang, 1992) from isotropic to anisotropic elasticity.

2. THE GOVERNING EQUATION

Let the z-axis be perpendicular to the transversely isotropic plane in a rectangular coordinate system (x, y, z). The generalized Hooke's law of a transversely isotropic body reads as

$$\left\{\begin{aligned} \sigma_x &= A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial v}{\partial y}+A_{13}\frac{\partial w}{\partial z},\\ \sigma_y &= A_{12}\frac{\partial u}{\partial x}+A_{11}\frac{\partial v}{\partial y}+A_{13}\frac{\partial w}{\partial z},\\ \sigma_z &= A_{13}\frac{\partial u}{\partial x}+A_{13}\frac{\partial v}{\partial y}+A_{33}\frac{\partial w}{\partial z},\end{aligned}\right. \qquad \left\{\begin{aligned} \tau_{yz} &= A_{44}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\\ \tau_{zx} &= A_{44}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\\ \tau_{xy} &= A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\end{aligned}\right. \tag{1}$$

where σ_x, $\sigma_y, \ldots, \tau_{xy}$ are stress components, u, v, w displacement components, A_{11}, $A_{12}, \ldots, A_{66}$ elastic constants and

$$2A_{66} = A_{11} - A_{12}. \tag{2}$$

The equations of equilibrium without a body force are

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = 0, \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = 0, \\ \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = 0. \end{cases} \tag{3}$$

The substitution of eqn (1) into eqn (3) yields the equation of equilibrium in terms of the displacement vector

$$\mathbf{Pu} = 0, \tag{4}$$

where $\mathbf{u} = (u, v, w)^{\mathrm{T}}$, (the superscript T denotes the transpose) and $\mathbf{P}$ is a 3×3 differential operator matrix

$$\mathbf{P} = \begin{bmatrix} \Lambda + \alpha_1 \dfrac{\partial^2}{\partial x^2} + \alpha_2 \dfrac{\partial^2}{\partial z^2} & \alpha_1 \dfrac{\partial^2}{\partial x \, \partial y} & \alpha_3 \dfrac{\partial^2}{\partial x \, \partial z} \\ \alpha_1 \dfrac{\partial^2}{\partial x \, \partial y} & \Lambda + \alpha_1 \dfrac{\partial^2}{\partial y^2} + \alpha_2 \dfrac{\partial^2}{\partial z^2} & \alpha_3 \dfrac{\partial^2}{\partial y \, \partial z} \\ \alpha_3 \dfrac{\partial^2}{\partial x \, \partial z} & \alpha_3 \dfrac{\partial^2}{\partial y \, \partial z} & \alpha_2 \Lambda + \alpha_4 \dfrac{\partial^2}{\partial z^2} \end{bmatrix}, \tag{5}$$

in which

$$\Lambda = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the two-dimensional Laplace operator and

$$\alpha_1 = \frac{A_{66} + A_{12}}{A_{66}}, \quad \alpha_2 = \frac{A_{44}}{A_{66}}, \quad \alpha_3 = \frac{A_{13} + A_{44}}{A_{66}}, \quad \alpha_4 = \frac{A_{33}}{A_{66}}, \tag{6}$$

The purpose of this paper is to find out the general solutions of eqn (4). For this reason, we introduce another 3×3 differential operator $\mathbf{Q}$, components of which are "algebraic complement minors" Q_{ij} of $\mathbf{P}$. They are in detail

$$\begin{cases} Q_{11} = \left(\Lambda + \alpha_1 \dfrac{\partial^2}{\partial y^2} + \alpha_2 \dfrac{\partial^2}{\partial z^2} \right) \left(\alpha_2 \Lambda + \alpha_4 \dfrac{\partial^2}{\partial z^2} \right) - \alpha_3^2 \dfrac{\partial^4}{\partial y^2 \, \partial z^2}, \\ Q_{22} = \left(\Lambda + \alpha_1 \dfrac{\partial^2}{\partial x^2} + \alpha_2 \dfrac{\partial^2}{\partial z^2} \right) \left(\alpha_2 \Lambda + \alpha_4 \dfrac{\partial^2}{\partial z^2} \right) - \alpha_3^2 \dfrac{\partial^4}{\partial x^2 \, \partial z^2}, \\ Q_{12} = Q_{21} = -\alpha_1 \left[\alpha_2 \Lambda + \left(\alpha_4 - \dfrac{\alpha_3^2}{\alpha_1} \right) \dfrac{\partial^2}{\partial z^2} \right] \dfrac{\partial^2}{\partial x \, \partial y}, \\ Q_{13} = Q_{11} = -\alpha_3 \nabla_0^2 \dfrac{\partial^2}{\partial x \, \partial z}, \\ Q_{23} = Q_{32} = -\alpha_3 \nabla_0^2 \dfrac{\partial^2}{\partial y \, \partial z}, \\ Q_{33} = (1 + \alpha_1) \nabla_0^2 \left(\Lambda + \beta \dfrac{\partial^2}{\partial z^2} \right), \end{cases} \tag{7}$$

where

$$\beta = \frac{\alpha_2}{1+\alpha_1} = \frac{A_{44}}{A_{11}}$$

and

$$\nabla_0^2 = \Lambda + \frac{1}{s_0^2}\frac{\partial^2}{\partial z^2}, \quad s_0^2 = \frac{1}{\alpha_2} = \frac{A_{66}}{A_{44}}. \tag{8}$$

The "determinant" of **P** is represented by $\mathscr{L}$,

$$\mathscr{L} = (1+\alpha_1)\alpha_2\nabla_0^2\nabla_1^2\nabla_2^2, \tag{9}$$

where

$$\nabla_i^2 = \Lambda + \frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}, \quad i = 1, 2, \tag{10}$$

in which s_1^2 and s_2^2 are two roots of the equation.

$$A_{33}A_{44}s^4 + (A_{13}^2 + 2A_{13}A_{44} - A_{11}A_{33})s^2 + A_{11}A_{44} = 0. \tag{11}$$

Lekhnitskii (1981, pp. 380–381) proved that the numbers s_1 and s_2 for any transversely isotropic body can be real or complex (with a real part different from zero), but cannot be pure imaginary.

From matrix algebra, it is easy to see

$$\mathbf{PQ} = \mathbf{QP} = \mathscr{L}\mathbf{I}, \tag{12}$$

where **I** is the 3×3 unit matrix.

3. THE GENERAL SOLUTIONS

The following theorem gives the general solutions of eqn (4).

Theorem 1

Assume that $\Psi = (\psi_1, \psi_2, \psi_3)^T$ satisfies the following equation

$$\mathscr{L}\Psi = \mathbf{0}. \tag{13}$$

Then

$$\mathbf{u} = \mathbf{Q}\Psi \tag{14}$$

is one of the solutions of the equation of equilibrium [eqn (4)] in terms of displacements.

Proof. Substituting eqn (14) into the left-hand side of eqn (4), we obtain

$$\mathbf{Pu} = \mathbf{PQ}\Psi = \mathscr{L}\Psi, \tag{15}$$

Equation (12) is used in the second part of eqn (15). From eqn (13), the right-hand side of eqn (15) is equal to the zero vector. In this way, eqn (14) is one of the solutions of eqn (4). The theorem is proved.

Lur'e (1937), Neuber (1964), Heki and Habara (1965) and Zhang (1979) had similar results with Theorem 1 in this paper. However, the following result is the new one. It shows the completeness of general solutions (14).

Theorem 2

Let **u** be an elastic displacement vector of eqn (4). Then there exists a vector function Ψ that satisfies eqns (13) and (14).

Proof. Assuming that **u** is a solution of eqn (4), let

$$\mathbf{a} = \frac{1}{(1+\alpha_1)\alpha_2}\mathscr{F}_0(\mathscr{F}_1(\mathscr{F}_2(\mathbf{u}))), \tag{16}$$

where $\mathscr{F}_j$ ($j = 0, 1, 2$) are generalized Newton potentials, i.e.

$$\mathscr{F}_j(f) = -\frac{s_j}{4\pi}\iiint \frac{f(\xi,\eta,\zeta)}{\rho_j}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \quad j = 0, 1, 2, \tag{17}$$

in which f is any function and

$$\rho_j = \sqrt{(x-\xi)^2+(y-\eta)^2+s_j^2(z-\zeta)^2}, \quad j = 0, 1, 2.$$

Because s_j^2 ($j = 0, 1, 2$) are not negative real numbers, the unique singular point in the volume integral of the right-hand side of eqn (17) is only (x, y, z). According to Newton's potential, we can still prove

$$\nabla_j^2\mathscr{F}_j(f) = f, \quad j = 0, 1, 2. \tag{18}$$

(Note: repeated indices do not imply summation in this paper.)

Operating $\mathscr{L}$ in both sides of eqn (16), we obtain

$$\mathscr{L}\mathbf{a} = \mathbf{u}. \tag{19}$$

Letting

$$\Psi = \mathbf{Pa}, \tag{20}$$

Ψ will satisfy the two requirements (13) and (14) of this theorem. In fact, from eqns (20) and (19), we have

$$\mathscr{L}\Psi = \mathbf{P}\mathscr{L}\mathbf{a} = \mathbf{Pu} = \mathbf{0}, \tag{21}$$

$$\mathbf{Q}\Psi = \mathbf{QPa} = \mathscr{L}\mathbf{a} = \mathbf{u}. \tag{22}$$

Equations (21) and (22) are eqns (13) and (14), respectively. This completes the proof of Theorem 2.

4. NONUNIQUENESS OF GENERAL SOLUTIONS

The potential functions ψ of general solutions (14) and (13) are nonunique.

Theorem 3

If some solution **u** of eqn (4) is represented in the same form as eqns (14) and (13), Ψ of eqns (14) and (13) may be changed into

$$\hat{\Psi} = \Psi + \mathbf{Ph}, \tag{23}$$

where $\mathbf{h} = (h_1, h_2, h_3)^T$ satisfies

$$\mathscr{L}\mathbf{h} = 0. \tag{24}$$

Proof. From eqns (14), (13), (23) and (24), we have

$$\mathbf{Q}\hat{\Psi} = \mathbf{Q}\Psi + \mathbf{QPh} = \mathbf{u} + \mathscr{L}\mathbf{h} = \mathbf{u}, \tag{25}$$

$$\mathscr{L}\hat{\Psi} = \mathscr{L}\Psi + \mathbf{P}\mathscr{L}\mathbf{h} = \mathbf{0}. \tag{26}$$

That is to say, $\hat{\Psi}$ satisfies the same equations as Ψ. So, the theorem is proved.
The following theorem represents the nondeterminate scope of Ψ.

Theorem 4
Assume that the solution $\mathbf{u}$ of (4) has two representations

$$\mathbf{u} = \mathbf{Q}\Psi_1, \tag{27}$$

$$\mathbf{u} = \mathbf{Q}\Psi_2, \tag{28}$$

Then there exists $\mathbf{h}$ which satisfies

$$\Psi_1 - \Psi_2 = \mathbf{Ph}, \tag{29}$$

where

$$\mathscr{L}\mathbf{h} = \mathbf{0}. \tag{30}$$

Proof. Subtracting eqn (28) from eqn (27), we obtain

$$\mathbf{Q}(\Psi_1 - \Psi_2) = \mathbf{0}. \tag{31}$$

Let

$$\mathbf{b} = \frac{1}{(1+\alpha_1)\alpha_2}\mathscr{F}_0(\mathscr{F}_1(\mathscr{F}_2(\Psi_1 - \Psi_2))). \tag{32}$$

Then

$$\mathbf{h} = \mathbf{Qb} \tag{33}$$

will satisfy eqns (29) and (30). In fact, we have

$$\mathbf{Ph} = \mathbf{PQb} = \mathscr{L}\mathbf{b} = \Psi_1 - \Psi_2,$$

$$\mathscr{L}\mathbf{h} = \mathbf{Q}\mathscr{L}\mathbf{b} = \mathbf{Q}(\Psi_1 - \Psi_2) = 0.$$

Thus Theorem 4 is proved.
Note: Theorems 1–4 can be extended to general anisotropic elastic bodies. Results of this kind will be given in greater detail in future papers.

5. LEKHNITSKII–HU–NOWACKI SOLUTIONS

Lekhnitskii–Hu–Nowacki solutions have forms

$$\begin{cases} u = \dfrac{\partial^2 F}{\partial x\,\partial z} - \dfrac{\partial \phi_0}{\partial y}, \\ v = \dfrac{\partial^2 F}{\partial y\,\partial z} + \dfrac{\partial \phi_0}{\partial x}, \\ w = -\alpha\left(\Lambda + \beta\dfrac{\partial^2}{\partial z^2}\right)F, \end{cases} \tag{34}$$

where

$$\alpha = \frac{A_{11}}{A_{13}+A_{44}}, \quad \beta = \frac{A_{44}}{A_{11}},$$

F and ϕ_0 satisfy the following equations, respectively,

$$\nabla_1^2\nabla_2^2 F = 0, \tag{35}$$

$$\nabla_0^2\phi_0 = 0. \tag{36}$$

It is not difficult to verify that eqns (34) are the solutions of eqn (4). It will be omitted.

In this section, we will prove that solutions (34) are complete if the elastic region Ω is z-convex.

From Theorem 2, we know that if $\mathbf{u}$ is a solution of eqn (4), there exists (ψ_1, ψ_2, ψ_3) such that

$$\begin{cases} u = Q_{11}\psi_1 + Q_{12}\psi_2 + Q_{13}\psi_3, \\ v = Q_{21}\psi_1 + Q_{22}\psi_2 + Q_{23}\psi_3, \\ w = Q_{31}\psi_1 + Q_{32}\psi_2 + Q_{33}\psi_3, \end{cases} \tag{37}$$

where Q_{ij} $(i, j = 1, 2, 3)$ are defined by eqn (7) and ψ_i $(i = 1, 2, 3)$ satisfy

$$\mathscr{L}\psi_i = 0, \quad i = 1, 2, 3. \tag{38}$$

From Theorem 3, ψ_i $(i = 1, 2, 3)$ in eqn (37) can be changed to $\hat{\psi}_i$ $(i = 1, 2, 3)$ as in eqn (23). Specially, taking $\mathbf{h} = \mathbf{k}h_3$ in eqn (23), we obtain

$$\begin{cases} \hat{\psi}_1 = \psi_1 + \alpha_3\dfrac{\partial^2 h_3}{\partial x\,\partial z}, \\ \hat{\psi}_2 = \psi_2 + \alpha_3\dfrac{\partial^2 h_3}{\partial y\,\partial z}, \\ \hat{\psi}_3 = \psi_3 + \left(\alpha_2\Lambda + \alpha_4\dfrac{\partial^2}{\partial z^2}\right)h_3, \end{cases} \tag{39}$$

where

$$\mathscr{L}h_3 = 0. \tag{40}$$

First we have:

Theorem 5

Assume that the elastic region Ω is z-convex and s_0^2, s_1^2, s_2^2 are not equal to each other. Then there exists h_3 such that

$$\hat{\psi}_i = \sum_{j=0}^{2} \hat{\psi}_i^{(j)}, \quad i = 1, 2. \tag{41}$$

where

$$\frac{\partial \hat{\psi}_1^{(j)}}{\partial x} + \frac{\partial \hat{\psi}_2^{(j)}}{\partial y} = 0, \quad j = 0, 1, 2, \tag{42}$$

$$\nabla_j^2 \hat{\psi}_i^{(j)} = 0, \quad i = 1, 2; \quad j = 0, 1, 2. \tag{43}$$

Proof. According to Lemma 4 of the Appendix, ψ_i ($i = 1, 2$) can be decomposed as

$$\psi_i = \sum_{j=0}^{2} \psi_i^{(j)}, \quad i = 1, 2, \tag{44}$$

where

$$\nabla_j^2 \psi_i^{(j)} = 0, \quad i = 1, 2; \quad j = 0, 1, 2. \tag{45}$$

From Lemma 2 of the Appendix, we know the following problems have solutions

$$\begin{cases} \dfrac{1}{s_j^2} \dfrac{\partial^3 h_3^{(j)}}{\partial z^3} = \dfrac{1}{\alpha_3} \left(\dfrac{\partial \psi_1^{(j)}}{\partial x} + \dfrac{\partial \psi_2^{(j)}}{\partial y} \right), \\ \nabla_j^2 h_3^{(j)} = 0, \end{cases} \quad j = 0, 1, 2. \tag{46}$$

Applying the second equation (46), we can rewrite eqn (46) as

$$\begin{cases} \Lambda \dfrac{\partial h_3^{(j)}}{\partial z} = -\dfrac{1}{\alpha_3} \left(\dfrac{\partial \psi_1^{(j)}}{\partial x} + \dfrac{\partial \psi_2^{(j)}}{\partial y} \right), \\ \nabla_j^2 h_3^{(j)} = 0, \end{cases} \quad j = 0, 1, 2. \tag{47}$$

Letting

$$\hat{\psi}_i^{(j)} = \psi_i^{(j)} + \alpha_3 \frac{\partial^2 h_3^{(j)}}{\partial x_i \, \partial z}, \quad i = 1, 2; \quad j = 0, 1, 2. \tag{48}$$

and using the first equation (47), we find

$$\frac{\partial \hat{\psi}_1^{(j)}}{\partial x} + \frac{\partial \hat{\psi}_2^{(j)}}{\partial y} = \frac{\partial \psi_1^{(j)}}{\partial x} + \frac{\partial \psi_2^{(j)}}{\partial y} + \alpha_3 \Lambda \frac{\partial h_3^{(j)}}{\partial z} = 0, \quad j = 0, 1, 2. \tag{49}$$

Equation (45) and the second equation (47) imply

$$\nabla_j^2 \hat{\psi}_i^{(j)} = \nabla_j^2 \psi_i^{(j)} + \alpha_3 \frac{\partial^2}{\partial x_i \partial z} \nabla_j^2 h_3^{(j)} = 0, \quad i = 1, 2; \quad j = 0, 1, 2, \tag{50}$$

where $x_1 = x$, $x_2 = y$.

Now we define h_3 by

$$h_3 = \sum_{j=0}^{2} h_3^{(j)}. \tag{51}$$

Then eqn (40) follows eqn (51) and the second equation (47). In view of eqns (48), (44) and (51), $\hat{\psi}_i$ $(i = 1, 2)$ admit the resolution of eqn (41); and eqns (49) and (50) are exactly equal to (42) and (43), respectively. This ends the proof of the theorem.

Next we come to prove:

Theorem 6

Assume that the elastic region Ω is z-convex and s_0^2, s_1^2, s_2^2 are not equal to each other. Then the Lekhnitskii–Hu–Nowacki solutions (34) are complete.

Proof. From Theorems 3 and 5, we know that if $\mathbf{u}$ is one of solutions of eqn (4), $\mathbf{u}$ can be represented as follows:

$$\begin{cases} u = Q_{11}\hat{\psi}_1 + Q_{12}\hat{\psi}_2 + Q_{13}\hat{\psi}_3, \\ v = Q_{21}\hat{\psi}_1 + Q_{22}\hat{\psi}_2 + Q_{23}\hat{\psi}_3, \\ w = Q_{31}\hat{\psi}_1 + Q_{32}\hat{\psi}_2 + Q_{33}\hat{\psi}_3, \end{cases} \tag{52}$$

where $\hat{\psi}_i$ $(i = 1, 2, 3)$ are given in eqns (39) and (51).

Let

$$A^{(j)} = \int_{\mathbf{x}_0}^{\mathbf{x}} \hat{\psi}_2^{(j)}\, \mathrm{d}x - \hat{\psi}_1^{(j)}\, \mathrm{d}y + B^{(j)}\, \mathrm{d}z. \tag{53}$$

$$B^{(j)} = \int_{\mathbf{x}_0}^{\mathbf{x}} \frac{\partial \hat{\psi}_2^{(j)}}{\partial z}\, \mathrm{d}x - \frac{\partial \hat{\psi}_1^{(j)}}{\partial z}\, \mathrm{d}y + s_j^2 \left(-\frac{\partial \hat{\psi}_2^{(j)}}{\partial x} + \frac{\partial \hat{\psi}_1^{(j)}}{\partial y} \right) \mathrm{d}z, \tag{54}$$

where $\mathbf{x}_0$ is some point of the region Ω and $\mathbf{x}$ is any point of the region Ω. Because of conditions (42) and (43), both linear integrals of eqns (53) and (54) are independent of routes.

Using eqns (53) and (54), one yields

$$\begin{cases} \dfrac{\partial}{\partial x} \nabla_j^2 A^{(j)} = \nabla_j^2 \hat{\psi}_2^{(j)} = 0, \\ \dfrac{\partial}{\partial y} \nabla_j^2 A^{(j)} = -\nabla_j^2 \hat{\psi}_1^{(j)} = 0, \quad j = 0, 1, 2. \\ \dfrac{\partial}{\partial z} \nabla_j^2 A^{(j)} = \nabla_j^2 B^{(j)} = 0, \end{cases} \tag{55}$$

This shows

$$\nabla_j^2 A^{(j)} = 0, \quad j = 0, 1, 2. \tag{56}$$

From eqn (53), we obtain

$$\hat{\psi}_1^{(j)} = -\frac{\partial A^{(j)}}{\partial y}, \quad \hat{\psi}_2^{(j)} = -\frac{\partial A^{(j)}}{\partial x}, \quad j = 0, 1, 2. \tag{57}$$

Letting

$$A = \sum_{j=0}^{2} A^{(j)}, \tag{58}$$

eqns (56) and (58) lead to

$$\mathscr{L}A = 0. \tag{59}$$

It follows from eqns (41), (57) and (58) that

$$\begin{cases} \hat{\psi}_1 = -\dfrac{\partial A}{\partial y}, \\ \hat{\psi}_2 = \dfrac{\partial A}{\partial x}. \end{cases} \tag{60}$$

Substitution of eqns (7) and (60) into eqn (52) and lengthy computation yields

$$\begin{cases} u = -(1+\alpha_1)\alpha_2\nabla_1^2\nabla_2^2\dfrac{\partial A}{\partial y} - \alpha_3\nabla_0^2\dfrac{\partial^2\hat{\psi}_3}{\partial x\,\partial z}, \\ v = (1+\alpha_1)\alpha_2\nabla_1^2\nabla_2^2\dfrac{\partial A}{\partial x} - \alpha_3\nabla_0^2\dfrac{\partial^2\hat{\psi}_3}{\partial y\,\partial z}, \\ w = (1+\alpha_1)\nabla_0^2\left(\Lambda + \beta\dfrac{\partial^2}{\partial z^2}\right)\hat{\psi}_3, \end{cases} \tag{61}$$

where

$$\begin{cases} \nabla_0^2\nabla_1^2\nabla_2^2 A = 0, \\ \nabla_0^2\nabla_1^2\nabla_2^2\hat{\psi}_3 = 0. \end{cases} \tag{62}$$

Let

$$\begin{cases} (1+\alpha_1)\alpha_2\nabla_1^2\nabla_2^2 A = \phi_0, \\ -\alpha_3\nabla_0^2\hat{\psi}_3 = F. \end{cases} \tag{63}$$

Then eqns (61) change into eqn (34). Therefore, solutions (34) are complete.

6. ELLIOTT–LODGE SOLUTIONS

The following solutions are from Elliott and Lodge.

$$\begin{cases} u = \dfrac{\partial}{\partial x}(\phi_1+\phi_2) - \dfrac{\partial \phi_0}{\partial y}, \\ v = \dfrac{\partial}{\partial y}(\phi_1+\phi_2) + \dfrac{\partial \phi_0}{\partial x}, \\ w = \dfrac{\partial}{\partial z}(k_1\phi_1 + k_2\phi_2), \end{cases} \tag{64}$$

where

$$k_i = \frac{A_{11} - A_{44}s_i^2}{(A_{13}+A_{44})s_i^2}, \quad i = 1, 2, \tag{65}$$

$$\nabla_i^2 \phi_i = 0, \quad i = 0, 1, 2. \tag{66}$$

Theorem 7

Assume that the elastic region Ω is z-convex and s_0^2, s_1^2, s_2^2 are not equal to each other. Then Elliott–Lodge solutions (64) are complete.

Proof. Firstly, it is easy to verify that eqns (64) are the solutions of eqn (4); so it is omitted. Secondly, we pointed out that any solution $\mathbf{u}$ of eqn (4) can be written in the form of eqns (64).

Suppose that $\mathbf{u}$ has been written in the form of eqns (34) since the Lekhnitskii–Hu–Nowacki solutions are complete.

According to Lemma 3 of the Appendix, we can assume that F of eqn (34) has the following resolution:

$$F = f_1 + f_2, \tag{67}$$

where

$$\nabla_i^2 f_i = 0, \quad i = 1, 2. \tag{68}$$

Setting

$$\phi_i = \frac{\partial f_i}{\partial z}, \quad i = 1, 2, \tag{69}$$

we have

$$\nabla_i^2 \phi_i = 0, \quad i = 1, 2. \tag{70}$$

Substitution of eqns (67) and (69) into eqns (34) yields

$$\begin{cases} u = \dfrac{\partial}{\partial x}(\phi_1+\phi_2) - \dfrac{\partial \phi_0}{\partial y}, \\ v = \dfrac{\partial}{\partial y}(\phi_1+\phi_2) + \dfrac{\partial \phi_0}{\partial x}, \\ w = -\alpha\left(\Lambda+\beta\dfrac{\partial^2}{\partial z^2}\right)(f_1+f_2). \end{cases} \tag{71}$$

Considering eqns (68) and (69) and the identities

$$\Lambda+\beta\frac{\partial^2}{\partial z^2} = \nabla_1^2 + \left(\beta-\frac{1}{s_1^2}\right)\frac{\partial^2}{\partial z^2} = \nabla_2^2 + \left(\beta-\frac{1}{s_2^2}\right)\frac{\partial^2}{\partial x^2}, \tag{72}$$

we have

$$-\alpha\left(\Lambda+\beta\frac{\partial^2}{\partial z^2}\right)(f_1+f_2) = \frac{\partial}{\partial z}(k_1\phi_1+k_2\phi_2), \tag{73}$$

where

$$k_i = -\alpha\left(\beta-\frac{1}{s_i^2}\right) = \frac{A_{11}-A_{44}s_i^2}{(A_{13}+A_{44})s_i^2}, \quad i = 1,2. \tag{74}$$

Substituting eqn (73) into eqns (71), we obtain eqns (64); and eqns (74) and (70) are exactly equal to eqns (65) and (66). Thus, solutions (64) are complete.

Note: the special cases that any two among s_0^2, s_1^2, s_2^2 are equal in theorems 5–7 will be studied in future papers.

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APPENDIX

A region Ω is called z-convex if any straight line parallel to the z-axis intersects the boundary of Ω in at most two points.

Lemma 1

If the region Ω is z-convex and f satisfies

$$\nabla_s^2 f = 0, \quad \text{in } \Omega, \tag{A1}$$

where s is not pure imaginary and

$$\nabla_s^2 = \Lambda + \frac{1}{s^2}\frac{\partial^2}{\partial z^2}, \tag{A2}$$

there exists A such that

$$\begin{cases} \nabla_s^2 A = 0, \\ \dfrac{\partial A}{\partial z} = f, \end{cases} \quad \text{in } \Omega. \tag{A3}$$

When $s = 1$, ∇_s^2 appears as the Laplace operator; Lemma 1 has been proved by Eubanks and Sternberg (1956). When $s \neq 1$, the proof is similar, so it is omitted.

Applying Lemma 1 several times, we have:

Lemma 2

In the same conditions as Lemma 1, there exists B such that

$$\begin{cases} \nabla_s^2 B = 0, \\ \dfrac{\partial^k B}{\partial z^k} = f, \end{cases} \quad \text{in } \Omega, \tag{A4}$$

where k are positive integers.

Lemma 3

If the region Ω is z-convex and

$$\nabla_1^2 \nabla_2^2 A = 0, \quad s_1^2 \neq s_2^2, \quad \text{in } \Omega, \tag{A5}$$

there exist $A^{(1)}$ and $A^{(2)}$ such that

$$A = A^{(1)} + A^{(2)}, \tag{A6}$$

$$\nabla_j^2 A^{(j)} = 0, \quad j = 1, 2. \tag{A7}$$

Proof. Letting

$$B = \nabla_2^2 A, \tag{A8}$$

then

$$\nabla_1^2 B = 0. \tag{A9}$$

According to Lemma 2, there exists $A^{(1)}$ such that

$$\begin{cases} \left(\dfrac{1}{s_2^2} - \dfrac{1}{s_1^2}\right) \dfrac{\partial^2 A^{(1)}}{\partial z^2} = B, \\ \nabla_1^2 A^{(1)} = 0. \end{cases} \tag{A10}$$

From eqns (A8) and (A10), we have

$$\nabla_2^2 A = \left(\frac{1}{s_2^2} - \frac{1}{s_1^2}\right) \frac{\partial^2 A^{(1)}}{\partial z^2} + \nabla_1^2 A^{(1)} = \nabla_2^2 A^{(1)}. \tag{A11}$$

Thus

$$\nabla_2^2 (A - A^{(1)}) = 0. \tag{A12}$$

Letting

$$A^{(2)} = A - A^{(1)}, \tag{A13}$$

(A12) becomes

$$\nabla_2^2 A^{(2)} = 0. \tag{A14}$$

The equation (A13) is the same as (A6), and eqn (A14) and the second equation (A10) are the same as eqns (A7); so the lemma is proved.

Similarly, we have:

Lemma 4

If the region Ω is z-convex and

$$\nabla_0^2 \nabla_1^2 \nabla_2^2 A = 0, \quad \text{in } \Omega,$$

where s_0^2, s_1^2, s_2^2 are not equal to each other, there exist $A^{(j)}$ ($j = 0, 1, 2$) such that

$$A = A^{(0)} + A^{(1)} + A^{(2)},$$

$$\nabla_j^2 A^{(j)} = 0, \quad j = 0, 1, 2.$$

Eubanks and Sternberg (1954) proved Lemmas 3 and 4 when the region Ω is a body of revolution.